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APPLICATION OF ISOTHERMAL SURFACES FOR CALCULATION OF UNSTEADY HEAT TRANSFER PROCESSES

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Consideration is given to a method of calculation of unsteady heat transfer processes using isothermal surfaces that are stationary in space, including time and, in the general case, three space coordinates. Conditions are obtained that must be satisfied by the equations of such surfaces.

In designing metallurgical units and developing a technology it is necessary to know the temperature distribution in equipment and lining components. Design methods and results of solutions are reported in [1-4]. For steady processes fixed isothermal surfaces exist, and if they are known, a solution can be obtained. In [5] conditions are determined that functions that are equations of a family of lines (or surfaces) must satisfy so that these lines (surfaces) can actually be isotherms.

In unsteady processes no fixed surfaces exist, as a rule, in the physical space. However, such stationary surfaces exist in a space-time continuum, i.e., in the space (four-dimensional in the general case) obtained by adding time to the space coordinates.

The equation of unsteady heat transfer in the curvilinear orthogonal coordinates α, β, γ is of the form [3, 4]

$$\frac{1}{H_1 H_2 H_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{\lambda H_2 H_3}{H_1}\frac{\partial t}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{\lambda H_1 H_3}{H_2}\frac{\partial t}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\frac{\lambda H_1 H_2}{H_3}\frac{\partial t}{\partial\gamma}\right)\right]=c\rho\frac{\partial t}{\partial\tau}. \quad (1)$$

For the two-dimensional problems $\partial t/\partial\gamma = 0$, $H_3 = 1$, while for problems with axial symmetry (with the axis of symmetry y) after introducing the coordinates $\alpha(x, y)$, $\beta(x, y)$ in a meridian plane [5] we obtain

$$\frac{\partial t}{\partial\gamma}=0\,,\quad H_3 = x(\alpha,\beta)\,.$$

If a function of coordinates and time $u(\alpha, \beta, \gamma, \tau)$ exists such that the temperature is only a function of u, i.e., $t(u)$, then the surfaces u = const are isothermal surfaces in the four-dimensional space $(\alpha, \beta, \gamma, \tau)$. Although the process is unsteady, these surfaces are stationary (in the space-time continuum).

To solve problems by the semi-inverse method, it is desirable to have a criterion for checking any differentiable functions $u(\alpha, \beta, \gamma, \tau)$ as to whether the corresponding surfaces can be isotherms (for steady-state problems this is condition (4) in [5]). With the assumption that t is a function of u, instead of (1) we obtain the equation

$$\frac{d^2 t}{du^2}+\varphi\frac{dt}{du}=0\,, \quad (2)$$

$$\varphi=\left[\frac{\lambda H_2 H_3}{H_1}\frac{\partial^2 u}{\partial\alpha^2}+\frac{\lambda H_1 H_3}{H_2}\frac{\partial^2 u}{\partial\beta^2}+\frac{\lambda H_1 H_2}{H_3}\frac{\partial^2 u}{\partial\gamma^2}+\frac{\partial u}{\partial\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\lambda H_2 H_3}{H_1}\right)+\right.$$

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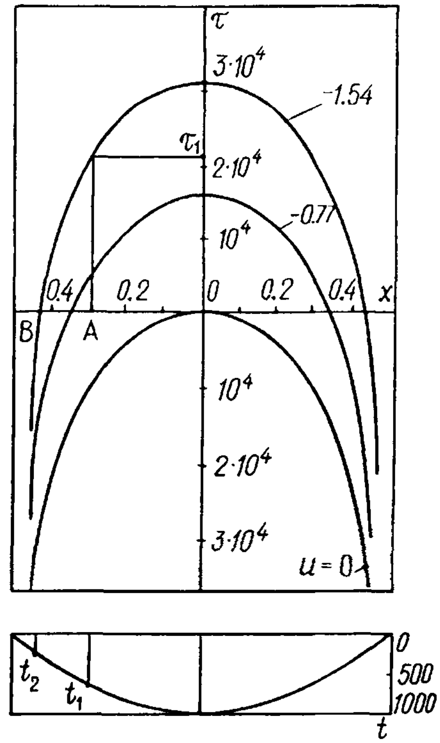

Fig. 1. Isotherms and initial temperature distribution of a line segment cooled with zero temperature of its ends. τ, sec; x, m; t, °C.

$$+\frac{\partial u}{\partial \beta}\frac{\partial}{\partial \beta}\left(\frac{\lambda H_1 H_3}{H_2}\right)+\frac{\partial u}{\partial \gamma}\frac{\partial}{\partial \gamma}\left(\frac{\lambda H_1 H_2}{H_3}\right)-c\rho H_1 H_2 H_3\frac{\partial u}{\partial \tau}\Bigg]\times$$

$$\times\left[\frac{\lambda H_2 H_3}{H_1}\left(\frac{\partial u}{\partial \alpha}\right)^2+\frac{\lambda H_1 H_3}{H_2}\left(\frac{\partial u}{\partial \beta}\right)^2+\frac{\lambda H_1 H_2}{H_3}\left(\frac{\partial u}{\partial \gamma}\right)^2\right]^{-1}. \qquad (3)$$

If φ can be represented as a function of u, then Eq. (2) has a solution (in a particular case, φ can be constant). If φ is not a function of u, then the assumption of isothermality of the surfaces u = const is incorrect.

As an example, we consider at first the case where the temperature depends only on one space coordinate when in cooling a segment of length $2l$ with the temperature equal to zero at $\alpha = x = \pm\, l$, $\partial t/\partial\beta = \partial t/\partial\gamma = 0$, $H_1 = H_2 = H_3 = 1$, the space-time continuum is two-dimensional and can be depicted graphically. Making the assumption that the isotherms can be described by the equation

$$u = m\tau + n\ln\left|\cos\left(\frac{\pi x}{2l}\right)\right|, \qquad (4)$$

where m, n are constants, we determine the following function from Eq. (3) with λ = const:

$$\varphi = \left(\frac{\partial^2 u}{\partial x^2}-\frac{c\rho}{\lambda}\frac{\partial u}{\partial \tau}\right)\left(\frac{\partial u}{\partial x}\right)^{-2} = -\frac{n\left(\frac{\pi}{2l}\right)^2+\frac{c\rho m}{\lambda}\cos^2\left(\frac{\pi x}{2l}\right)}{n^2\left(\frac{\pi}{2l}\right)^2\sin^2\left(\frac{\pi x}{2l}\right)}.$$

For $m = -(\lambda n/c\rho)(\pi/2l)^2$ the function $\varphi = \text{const} = -1/n$, and Eq. (2) has the solution

$$t(u) = C_1 + C_2\exp\left(\frac{u}{n}\right), \qquad (5)$$

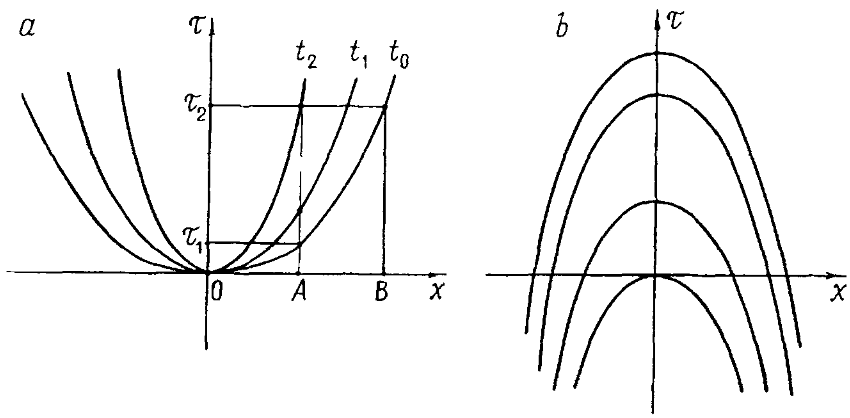

Fig. 2. Isotherms in the form of parabolas: a) with a common point of all theparabolas at the origin of coordinates (the Stefan problem); b) with a shift of the parabolas relative to each other along the τ-axis.

$$u = n\left[-\frac{\lambda}{c\rho}\left(\frac{\pi}{2l}\right)^2 \tau + \ln\left|\cos\left(\frac{\pi x}{2l}\right)\right|\right],$$

where C_1, C_2 are integration constants determined by the boundary conditions.

Figure 1 shows isotherms in the coordinates x, τ constructed for $C_1 = 0$, $C_2 = t_{in} = 1000°C$ for a segment $l = 0.5$ m. Here t_{in} is the initial temperature of the middle of the segment ($x = 0$, $\tau = 0$), and the curve $t(x)$ at $\tau = 0$ is represented in Fig. 1. The calculations are made for steel with $\lambda = 25$ W/(m·deg), $c = 710$ J/(kg·deg), $\rho = 7800$ kg/m^3, $2l = 1$ m, $n = 1$, $u = -5.16 \cdot 10^{-5}\tau + \ln|\cos(3.14x)|$.

Figure 1 shows three isotherms: for $u = 0$, $t = t_{in} = 1000°C$; along the second isotherm $u = -0.77$, $t = 463°C$; for $u = -1.54$, $t = 213°C$. These curves are actually isotherms (lines of constant temperature), but in a system of space-time coordinates. For the ends of the segment the isotherms are straight lines parallel to the τ axis, $x = \pm 0.5$ m (here $t = 0$). At the initial moment the point A of the segment had the temperature t_1, and at the moment of time τ_1 it passed (in moving along a straight line parallel to the τ axis) to the isotherm $u = -1.54$, and the temperature at it was equal to the temperature at the point B at the initial moment, i.e., to t_2.

It is easy to verify that in the two-dimensional (x, τ) problem isotherms can be: straight lines

$$u = k\tau - x, \tag{6}$$

parabolas passing through the point $x = \tau = 0$

$$u = \frac{x^2}{k\tau} \tag{7}$$

and parabolas obtained by a shift relative to each other along the τ axis

$$u = k\tau + x^2, \tag{8}$$

where k is a constant.

Using (6), we obtain $\varphi = -c\rho k/\lambda$, and from formula (7) it follows that

$$\varphi = \frac{k\tau}{2x^2} + \frac{c\rho k}{4\lambda} = \frac{1}{2u} + \frac{c\rho k}{4\lambda}. \tag{9}$$

This yields the well-known Stefan solution for crystallization expressed in terms of the Laplace function [1-3, 6]. After substituting (8) into (3) we arrive at

$$\varphi = \left(2 - \frac{c\rho k}{\lambda}\right) \frac{1}{4x^2}$$

and solution (2) exists only for $k = 2\lambda / c\rho$, i.e.,

$$u = x^2 + \frac{2\lambda}{c\rho}\tau\,, \quad \varphi = 0\,,$$
$$t(u) = C_1 + C_2 u\,. \tag{10}$$

Figure 2 shows isotherms in the form of parabolas, and in conformity with (7) Fig. 2a depicts lines t = const passing through the origin of the coordinates $x = \tau = 0$, while Fig. 2b shows parabolas corresponding to (8) shifted relative to each other along the τ axis. The scheme in Fig. 2a corresponds to the Stefan problem [6] of crystallization of a liquid phase (without its superheating) at a constant temperature t_0 starting from the plane x = 0. The line t_0 = const, $u = u_0$ is the boundary of the phases, with the liquid phase being to the right of it (Fig. 2a) and the solid state to the left of it. For instance, in the cross section A solidification started at the moment τ_1 when the segment OA was equal to the thickness of the solid phase $\delta = \sqrt{u_0 k \tau_1}$. At the moment τ_2 the temperature of the point A was equal to τ_2, while the thickness of the solid phase corresponded to the segment OB. The solution of the Stefan problem is well studied and is consistent with experimental data for a constant temperature of the surface of the hardened body [6]. According to Eqs. (6), (7), the isotherms in the coordinates x, τ can be both parabolas and straight lines. Both these solutions can be used to describe the solidification process with a constant temperature at the movable phase boundary. Let us compare these two solutions. In the known Stefan solution $u = x^2/k\tau$ the condition of temperature constancy is fulfilled on a surface whose motion is determined by the condition x = const $\sqrt{\tau}$ (the so-called "square-root law," see [6]). For the straight isotherms $u = k\tau - x$ the condition of temperature constancy is fulfilled for $x = k\tau$ − const, i.e., the temperature at the movable boundary is constant but for a law of motion of the liquid-solid interface other than that in the Stefan problem. If at the initial moment $\tau = 0$, $x = 0$, then the thickness of the hardened layer is equal to $\delta = \sqrt{k\tau}$ in the Stefan solution and $\delta = k\tau$ in the new solution. Here, for the new solution φ = const = $-c\rho k/\lambda$, and the exact solution of heat conduction equation (2) is of the form

$$t(u) = C_1 + C_2 \exp\left(\frac{c\rho k}{\lambda} u\right), \tag{11}$$

while in the Stefan problem the analogous solution is determined by the Laplace function ("the error function") [6].

At the phase boundary the heat balance condition must also be fulfilled, which for constancy of the temperature of the liquid phase, i.e., in the absence of superheating, has the form

$$\lambda \frac{\partial t}{\partial x} = \rho L \frac{d\delta}{d\tau}$$

for $u = u_0$ = const, where $t = t_0$ = const. These boundary conditions determine the constants C_1, C_2 in formula (11), and the solution obtained is

$$t(u) = t_0 - \frac{L}{c}\left[\exp\left(\frac{c\rho k}{\lambda} u\right) - 1\right].$$

The well-studied Stefan solution [6] and the solution corresponding to formula (11) satisfy the same boundary conditions at the movable phase boundary, but the boundary conditions on the second surface, for instance, on the fixed surface x = const = 0, are different for them. In the Stefan solution $t(0)$ = const at $x = u = 0$ and solidification occurs in this case at a constant temperature of the surface $x = 0$. But for the solution (11), at $x = 0$, $u = k\tau$

$$t(0) = t_0 - \frac{L}{c}\left[\exp\left(\frac{c\rho k^2}{\lambda}\tau\right) - 1\right], \tag{12}$$

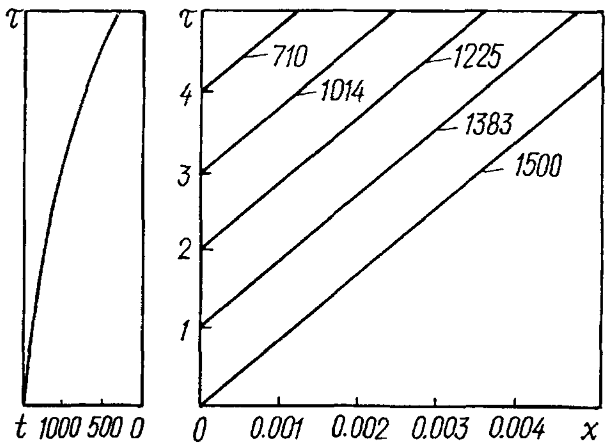

Fig. 3. Isotherms and the change in the surface temperature in solidification with a constant velocity of the phase boundary.

and the heat flux on this surface is

$$q(0) = \rho k L \exp\left(\frac{c\rho k^2}{\lambda}\tau\right), \quad k = \frac{q_0}{L\rho}.$$

The solution (11) can be used only in the case where the boundary conditions are close to those determined by formula (12) or can be matched with it owing to a proper choice of the constant k.

Depending on the cooling conditions of the surface of the hardening body, an acceptable solution can be chosen. If the surface temperature changes slightly and can be taken to be constant, the Stefan solution is definitely preferable. However, for intense cooling of the surface and a rapid decrease in its temperature (which is often the case with continuous metal casting) it is preferable to use formulas (11), (12).

The constant k is determined by the boundary conditions; for instance, if a heat flux q_0 is specified at $x = \tau = 0$, then

$$k = \frac{q_0}{\rho L}, \quad q(0) = k_0 \exp\left(\frac{c q_0^2}{\rho \lambda L^2}\tau\right).$$

These formulas were used in a calculation of temperature regimes of continuous casting of thin 60 × 1200 mm slabs at high rates up to 3.5 m/min. In this case cooling proceeded at a high rate both in the short crystallizer and under the latter, where sprayers for cooling the slab surface were installed. Calculations were made for carbon steel with $\rho = 7800$ kg/m^3, $c = 710$ J/(kg·deg), $\lambda = 29$ W/(m·deg), $t = 1500$°C, $L = 2.68 \cdot 10^5$ J/kg, $q_0 = 2.5 \cdot 10^6$ W/m^2, $k = 2.5 \cdot 10^6/(7800 \cdot 2.68 \cdot 10^5) = 1.2 \cdot 10^{-3}$ m/sec. The solidification rate was equal to 1.2 mm/sec, which agrees with experimental data for the initial stage of crystallization of steels. The change in the surface temperature ($x = 0$) is shown in Fig. 3, where isotherms (with indices equal to the temperatures) are also given. The isotherm $t_0 = 1500$°C restricts the solid phase (unlike the parabola in the case of the Stefan solution). This solution for the case of a constant rate of crystallization requires rapid cooling of the surface with boundary conditions corresponding to (12). When the Stefan solution with a constant surface temperature is used, the motion of the phase boundary (and the thickness of the solid layer) is described by the "square-root law," and the velocity of this boundary decreases monotonically with increasing thickness of the hardened layer. The obtained solution (11) characterizes motion of the phase boundary with a constant velocity, and in order to ensure such a crystallization mode, it is necessary to provide higher-rate cooling of the boundary surface of the solid in conformity with formula (12). The isotherms demonstrate the temperature distribution in the plane x, τ.

For polar coordinates in the two-dimensional problem $x = \alpha \cos\beta$, $y = \alpha \sin\beta$, $H_1 = 1$, $H_2 = \alpha$, $H_3 = 1$, and the linear function (6) cannot represent an isotherm. However, the quantity $u = n(\tau + (c\rho/4\lambda)\alpha^2)$ similar to (8) gives $\varphi = 0$ and a solution in the form of (10). The variable (7) also provides the dependence $\varphi(u)$, and consequently, the parabolic dependences $\tau(\alpha)$ determine isotherms in polar coordinates as well. These results can be extended to the case of an asymmetric distribution of temperatures. If isotherms are adopted in the form

$$u = \tau + n\beta + f(\alpha),$$

where n is a constant, $f(\alpha)$ is a function of the variable α, then $\varphi = 0$ for $f(\alpha) = \ln \alpha + c\rho\alpha^2/4\lambda$.

Let us consider curvilinear orthogonal coordinates that are supplemented with time as an additional coordinate. To calculate parabolic coordinates in the two-dimensional problem [4], we take

$$x = \alpha\beta, \quad y = 0.5\,(\beta^2 - \alpha^2), \quad H_1 = H_2 = \sqrt{\alpha^2 + \beta^2}, \quad H_3 = 1.$$

It is easy to check that $\varphi = 0$ for the variables

$$u = \alpha + \beta + k\tau + \frac{c\rho k}{12\lambda}(\alpha^4 + \beta^4), \quad u = \exp(-k\tau)\,J_0\left[\sqrt{\left(\frac{c\rho k}{4\lambda}\right)}\,(\alpha^2 + \beta^2)\right], \tag{13}$$

where k is a constant that depends on the boundary conditions; J_0 is a Bessel function of zeroth order.

Both formulas of (13) lead to a solution in the form of (10). If the boundary conditions correspond to the experimental data with sufficient accuracy here, then the solution is acceptable. These results were used for temperature calculation of inserts in the form of parabolic cylinders. Such inserts on a continuous casting machine of the "Azovstal'" metallurgical works were used in crystallizers for molding depressions on slab surfaces. After leaving the crystallizer, each of wide faces of the slab had two depressions of approximately parabolic form. Then the slab was cut along these depressions into individual billets by oxygen cutting for the slab rolling mills. For the axisymmetric problem in parabolic coordinates it is necessary to take $H_3 = \alpha\beta$, and then the following solutions can be obtained:

$$u = k\tau + n \ln\alpha \ln\beta + \frac{c\rho k}{16\lambda}(\alpha^4 + \beta^4),$$

$$u = \frac{\exp(-k\tau)}{\sqrt{\alpha^2 + \beta^2}}\,J_{1/2}\left[\sqrt{\left(\frac{c\rho k}{4\lambda}\right)}\,(\alpha^2 + \beta^2)\right] =$$

$$= \frac{2\exp(-k\tau)}{\alpha^2 + \beta^2}\sqrt{\left(\frac{1}{\pi}\sqrt{\left(\frac{\lambda}{c\rho k}\right)}\right)}\,\sin\left[\sqrt{\left(\frac{c\rho k}{4\lambda}\right)}\,(\alpha^2 + \beta^2)\right],$$

where $J_{1/2}$ is a Bessel function of the first kind of order 0.5, and in both cases $\varphi = 0$ and formula (10) determines the solution.

Equations (13) with the condition $u = \text{const}$ determine isothermal surfaces in the coordinates α, β, τ.

If the coordinates in the plane are logarithmic spirals, then $H_1 = \sqrt{\beta/2\alpha}$, $H_2 = \sqrt{\alpha/2\beta}$, $H_3 = 1$ (see [4]), and solution (10) for $\varphi = 0$ is realized for isothermal surfaces:

$$u = \frac{c\rho k}{4\lambda}\alpha\beta + k\tau, \quad u = \exp(-k\tau)\,J_0\left(\sqrt{\left(\frac{c\rho k}{\lambda}\alpha\beta\right)}\right).$$

Not only in Cartesian but also in a number of curvilinear coordinate systems there are isotherms that are determined by the product of an exponential function of the time and some function of the coordinates.

In the bipolar coordinates

$$x^2 + (y - \operatorname{cotan}\alpha)^2 = 1 + \operatorname{cotan}^2\alpha,$$

$$(x - \coth\beta)^2 + y^2 = \coth^2\beta - 1, \tag{14}$$

$H_1 = H_2 = 1/(\cosh\beta - \cos\alpha)$ (see [4, 5]) use can also be made of formula (10) by applying the following isotherms for the two-dimensional problem:

$$u = k\tau + \frac{c\rho k}{2\lambda} \frac{\mathrm{sh}^2\beta}{(\cosh\beta - \cos\alpha)^2}, \quad \varphi = 0 .$$

For the axisymmetric problem (cooling of a body in the form of a torus with mean radius $\coth\beta$ and radius of the generating curcle $\sqrt{\coth^2\beta - 1}$) $\varphi = 0$ and solution (10) occurs for the isotherms

$$u = k\tau + \frac{c\rho k}{2\lambda} \frac{\sin^2\alpha}{(\cosh\beta - \cos\alpha)^2} .$$

In problems solved in a space of i measurements ($i = 1, 2, 3$) unsteady processes can fail to have fixed isotherms. But such isothermal surfaces exist in the space of $(i + 1)$ measurements in which time is the additional coordinate. In solving problems by the semi-inverse method the choice of suitable isothermal surfaces with the aid of the function φ allows the problem to be reduced in a number of cases to the solition of an ordinary differential equation of the second order.

NOTATION

c, heat capacity of the material; f, function of the variable α; H_1, H_2, H_3, coefficients of the first square form (Lamé); k, constant characterizing the cooling rate; L, latent heat of solidification; l, half-length of the segment, the temperature of whose ends is equal to zero; m, n, constants characterizing the isothermal surface, see (4); q, heat flux; $q(0)$, heat flux on the surface of the body ; q_0, heat flux on the surface of the body at the moment of the beginnin of cooling ($\tau = 0$); t, temperature; t_{in}, initial temperature at $\tau = 0$; t_0, solidification temperature; t_1, t_2, temperatures at the moments of time τ_1 and τ_2; u, quantity determining the isothermal surface in the space-time continuum; u_0, quantity u at the solidification temperature t_0; x, y, z, Cartesian coordinates; α, β, γ, curvilinear orthogonal coordinates; δ, thickness of the material hardened under crystallization conditions; λ, thermal conductivity; ρ, density of the material; φ, function determined by (3); τ, time.

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